ITERATION OF MAPS BY PRIMITIVE SUBSTITUTIVE SEQUENCES

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Abstract. Let $A$ and $F$ be finite sets and suppose for each $a \in A$ we have a map $\varphi_a : F \to F$. Suppose $\omega = \omega_1\omega_2\omega_3\ldots$ is a sequence in $A$ which is ultimately primitive substitutive, i.e., a tail of $\omega$ is the image (under a letter to letter morphism) of a fixed point of a primitive substitution. We show that the induced sequence of iterates $(\varphi_{\omega_n} \circ \varphi_{\omega_{n-1}} \circ \cdots \circ \varphi_{\omega_1})_{n=1}^{\infty}$ is also ultimately primitive substitutive, and is itself primitive substitutive if it is recurrent.

1. Introduction and Definitions

Let $A$ be a finite nonempty set. J.-P. Allouche and M. Mendes France[1] showed that if $\star$ is an associative binary operation on $A$ and $\omega \in A^n$ a $q$-automatic sequence, then the sequence of partial products $(\omega_n \star \omega_{n-1} \star \cdots \star \omega_1)_{n=1}^{\infty}$ is also $q$-automatic.\(^1\) The result was used by S. Lehr to show that the set $M(q,r)$ of real numbers whose fractional part has a $q$-automatic base $r$ digit expansion, is a $\mathbb{Q}$-vector space.[8] It was generalized by F.M. Dekking as follows. Let $A$ and $F$ be finite sets, and suppose that for each letter $a \in A$ we have a map $\varphi_a : F \to F$. If $\omega = \omega_1\omega_2\omega_3\ldots$ is a substitutive sequence in $A^\mathbb{N}$, then the induced sequence of iterates $(\varphi_{\omega_n} \circ \varphi_{\omega_{n-1}} \circ \cdots \circ \varphi_{\omega_1})_{n=1}^{\infty}$ is also substitutive.[3] In this note we obtain further results of this type for primitive substitutive sequences. Before stating our main theorem we recall some definitions.

A substitution on the alphabet $A$ is a map $\tau : A \to A^+$, where $A^+$ is the set of all nonempty words in $A$, i.e., $A^+ = \bigcup_{n \geq 1} A^n$. The morphism $\tau$ extends by concatenation to morphisms from $A^+$ to $A^+$ and from $A^n$ to $A^m$. A substitution $\tau$ is primitive if there exists $n \in \mathbb{N}$ such that for all $a, b \in A$ the word $\tau^n(a)$ contains an occurrence of $b$. We call a sequence $\omega = \omega_1\omega_2\omega_3\ldots \in A^\mathbb{N}$ (primitive) substitutive if it is the image under a letter-to-letter morphism of a fixed point of a (primitive) substitution; i.e., if there exists a finite alphabet $B$ and a morphism $\pi : B \to A$ such that $\omega = \pi(\omega')$ where $\omega' \in B^\mathbb{N}$ is a fixed point of $\pi$ (primitive) substitution on $B$. We say $\omega$ is ultimately primitive substitutive (or u.p.s.) if a tail of $\omega$ is primitive substitutive, i.e., if $\omega_n\omega_{n+1}\omega_{n+2}\ldots$ is primitive substitutive for some $n \geq 1$. A sequence $\omega$ is said to be recurrent if every subword of $\omega$ occurs an infinite number of times in $\omega$, uniformly recurrent (or minimal) if each subword occurs with bounded gaps in $\omega$, and linearly recurrent[4] if there is some constant $c$ such that the maximum gap between consecutive occurrences of any subword of length $n$ is less than or equal to $cn$. A primitive substitutive sequence is linearly recurrent.[5]

\(^{1}\)J.-P. Allouche (personal communication) remarks that this result may be inferred from the work of Cobham (1972).
Let \( A \) and \( F \) be finite sets. Suppose that for each letter \( a \in A \) we have a map \( \varphi_a : F \to F \). Let \( \omega = \omega_1\omega_2\omega_3 \ldots \in A^\mathbb{N} \). We consider the induced sequence of iterates \((\varphi_{\omega(n)})_{n=1}^\infty \) where \( \varphi_{\omega(n)} = \varphi_{\omega_n} \circ \varphi_{\omega_{n-1}} \circ \cdots \circ \varphi_{\omega_1} \in F^F \). Our main result is:

**Theorem 1.** If \( \omega \) is u.p.s. then so is \((\varphi_{\omega(n)})_{n=1}^\infty \). Moreover, if \((\varphi_{\omega(n)})_{n=1}^\infty \) is recurrent then it is primitive substitutive (in particular linearly recurrent).

We actually prove a slightly stronger result, that the sequence \((\varphi_{\omega(n)}, \omega_n)_{n=1}^\infty \) is u.p.s.. Although the statement of Theorem 1 is a slight variant of Dekking’s result, to our knowledge its proof cannot be deduced from Dekking’s proof. In fact, the first statement of Theorem 1 is false if u.p.s. is replaced by primitive substitutive since the induced sequence of iterates need not be recurrent. Our proof of Theorem 1 relies on the following key lemma which may be of independent interest:

**Lemma 2.** Let \( X \in A^\mathbb{N} \) be primitive substitutive. Then the shifted sequence \( T X \) is primitive substitutive.\([4]\) If \( wX \) is recurrent \((w \in A^+)\) then \( wX \) is primitive substitutive.

In other words, Lemma 2 asserts that in a minimal symbolic dynamical system \((X, T)\), any point in \( X \) in the orbit of a primitive substitutive sequence is primitive substitutive. The second statement of Lemma 2 uses a characterization of primitive substitutive sequences discovered independently by F. Durand\([5]\) and by the authors\([6]\).

We end the paper with a few examples.

2. **Proof of Theorem 1**

We begin by establishing some notation which will be used in the proof of Theorem 1. For \( w \in A^n \) and \( 1 \leq i \leq n \) put \( |w| = n \) (the length of \( w \)), \( w_i \) is the \( i \)th letter of \( w \) (so that \( w = w_1w_2 \ldots w_n \)) and \( w(i) = w_1w_2 \ldots w_i \) (the prefix of \( w \) of length \( i \)). It is convenient to introduce the ‘empty word’ \( \varepsilon \) of length 0 with the understanding that \( \tau(\varepsilon) = \varepsilon \), \( w(0) = \varepsilon \) and \( \varphi_\varepsilon = 1_F \) (the identity map \( F \to F \)). For a sequence \( \omega \in A^\mathbb{N} \) and \( i \in \mathbb{N} \) we write \( \omega_i \) for the \( i \)th letter of \( \omega \) and \( \omega(i) = \omega_1\omega_2 \ldots \omega_i \). Let \( T : A^\mathbb{N} \to A^\mathbb{N} \) denote the left shift map, i.e., \( T \omega_1\omega_2\omega_3 \ldots = \omega_2\omega_3\omega_4 \ldots \).

**Proof of Theorem 1.** Let \( A \) and \( F \) be finite sets, and suppose that for each letter \( a \in A \) we have a map \( \varphi_a : F \to F \). For \( w \in A^+ \) set \( \varphi_w = \varphi_{w_1} \circ \varphi_{w_{|w|-1}} \circ \cdots \circ \varphi_{w_1} \). Let \( \omega = \omega_1\omega_2\omega_3 \ldots \in A^\mathbb{N} \) be u.p.s.. We may assume that \( \omega \) itself is primitive substitutive. Indeed, if \((\varphi_{\omega_n} \circ \varphi_{\omega_{n-1}} \circ \cdots \circ \varphi_{\omega_1})_{n=1}^\infty \) is u.p.s. then for any map \( \varphi : F \to F \) the sequence \((\varphi_{\omega_n} \circ \varphi_{\omega_{n-1}} \circ \cdots \circ \varphi_{\omega_1} \circ \varphi)_{n=1}^\infty \) is also u.p.s.. We may further assume that \( \omega \) is a fixed point of a primitive substitution \( \tau \) on \( A \), for if \( \omega = \pi(\omega') \) where \( \omega' \) is a fixed point of a primitive substitution on \( B \) and \( \pi : B \to A \), then setting \( \varphi_b := \varphi_{\pi(b)} \) \((b \in B)\) yields the same sequence of iterates for \( \omega' \).

We will show that for suitably chosen \( k \in \mathbb{N} \), the sequence

\[
\gamma = ((\varphi_{\tau^k(\omega(n-1))), \omega_n}))_{n=1}^\infty
\]

in \( B = F^F \times A \) is u.p.s.. Our method of proof is to produce a primitive substitution on a subalphabet \( B' \) of \( B \) fixing a sequence a tail of which is also a tail of \( \gamma \). We may then apply Lemma 2 above to conclude that \( \gamma \) is u.p.s.. The theorem follows from the other implication of Lemma 2 and Proposition 3.1 of Durand (1998), since we can obtain \((\varphi_{\omega(n)})_{n=1}^\infty \) from \( \gamma \) by the morphism

\[
(\varphi, a) \mapsto \varphi_{\tau^k(a)(1)} \circ \varphi, \varphi_{\tau^k(a)(2)} \circ \varphi, \ldots, \varphi_{\tau^k(a)} \circ \varphi.
\]
Choose nonnegative integers \(k\) and \(p\) such that \(\varphi_{r+k}(a) = \varphi_{r+k+p}(a)\) holds for every \(a \in A\), and define a substitution \(\zeta\) on \(B\) by setting

\[
\zeta(\varphi, a) = (\varphi, \tau^{p}(a_1)) (\varphi_{r+k}(\tau^{p}(a_1))) \circ \varphi, \tau^{p}(a_2) \ldots
\]

\[
\ldots (\varphi_{r+k}(\tau^{p}(a_{|\tau^{p}(a)-1|})) \circ \varphi, \tau^{p}(a_{|\tau^{p}(a)|})).
\]

It is easy to see that for every \(\varphi \in F^F\), the sequence \((\varphi_{r+k}(\omega(n-1)) \circ \varphi, \omega_n)_{n=1}^{\infty}\) is fixed by \(\zeta\). Let \(\gamma\) be the sequence fixed by \(\zeta\) which begins in \((1_F, \omega_1)\). Thus,

\[
\gamma = (1_F, \omega_1)(\varphi_{r+k}((\omega_1)), \omega_2)(\varphi_{r+k}((\omega_2)), \omega_3)(\varphi_{r+k}((\omega_3)), \omega_4) \ldots
\]

For each letter \((\varphi, a) \in B\) which occurs in \(\gamma\), let \(B_{(\varphi,a)}\) be the subalphabet of \(B\) consisting of all letters which occur in some \(\zeta^n((\varphi, a))\), \(n = 1, 2, \ldots\). Note that every letter of each \(B_{(\varphi,a)}\) occurs in \(\gamma\). Let \(B'\) be a \(B_{(\varphi,a)}\) with the smallest cardinality and let \(\zeta'\) be the restriction of \(\zeta\) to \(B'\). It follows from the definition of \(\zeta\) and primitivity of \(\tau\) that \(B'\) contains some letter of the form \((\psi, \omega_1)\). Thus \(\zeta'\) has a fixed point in \((B')^n\); this together with the minimality of \(B'\) implies that \(\zeta'\) is a primitive substitution. Fix \(\psi \in F^F\) such that \((\psi, \omega_1) \in B'\) and let \(\zeta'\) be the sequence beginning in \((\psi, \omega_1)\) fixed by \(\zeta'\). If the \(n\)th letter of \(\gamma\) is \(\gamma_n = (\varphi^{(n)}, \omega_n)\) then the \(n\)th letter of \(\zeta'\) is

\[
\gamma_n' = (\varphi^{(n)} \circ \psi, \omega_n).
\]

Let \(l < m\) be positive integers such that \(\psi^l = \psi^m\). Now \((\psi, \omega_1)\) must occur in \(\gamma\) because every letter of \(\zeta'\) occurs in \(\gamma\). Thus in view of (1), \((\psi^2, \omega_1)\) must occur in \(\gamma'\), hence also in \(\gamma\). Continuing in this manner one finds that every letter \((\psi^r, \omega_1)\) occurs in \(\gamma\) and \(\gamma'\). In particular there is a positive integer \(N\) such that \(\gamma_N = (\psi^l, \omega_l)\) (and \(\gamma_N' = (\psi^{l+1}, \omega_1)\)). The sequence \(\gamma'' := (\varphi^{(n)} \circ \psi \circ \psi^{m-l-1}, \omega_n)\) is fixed by the primitive substitution \(\zeta'\) and \(\gamma''_n = \gamma_n\) for all \(n \geq N\). The theorem follows from the previous remarks.

3. Proof of Lemma 2

Durand showed that \(TX\) is primitive substitutive.[4] We recall some definitions from Durand (1998). Suppose \(Y \in A^\omega\) is uniformly recurrent and let \(u\) be a nonempty prefix of \(Y\). Denote by \(H_u(Y)\) the set of return words of \(Y\) to \(u\), i.e., those words \(v\) which occur in \(Y\) such that

- \(vu\) occurs in \(Y\)
- \(v\) is a prefix and a suffix of \(vu\) and has no other occurrence in \(vu\).

The sequence \(Y\) can be written in a unique way as a concatenation

\[
Y = p_1p_2p_3 \ldots , \quad p_i \in H_u(Y).
\]

Give \(H_u(Y)\) the linear order defined by the rank of first occurrence in this sequence. This defines a bijection \(\Lambda_{Y,u} : H_u(Y) \rightarrow \{1, 2, \ldots, \text{Card}(H_u(Y))\}\), and the sequence

\[
D_u(Y) = \Lambda_{Y,u}(p_1)\Lambda_{Y,u}(p_2) \ldots
\]

is called a derived sequence of \(Y\). Durand showed that a sequence \(Y\) is primitive substitutive if and only if its set of derived sequences is finite.[5]

As a primitive substitutive sequence, \(X\) is uniformly recurrent, and since \(Y = wX\) is recurrent, \(X\) is also uniformly recurrent. For every sufficiently long prefix \(u\) of \(X\) we have \(\min\{|v| : v \in H_u(X)\} > |w|\) (unless \(X\) is a periodic sequence, in which case \(Y\) is also periodic and hence primitive substitutive). We will show that the number of derived sequences of \(Y\) obtained from the prefixes \(wu\), with \(u\) as above, is finite.
With $u$ as above write

$$X = p_{u,1}p_{u,2} \ldots, \quad p_{u,i} \in H_u(X).$$

Since $wX$ is recurrent there must be some $p_{u,0} \in H_u(X)$ having $w$ as a suffix such that $p_{u,0}X$ is recurrent. Set

$$L_{u,w}(X) = \{ p_u : p_u \in H_u(X) \text{ and } w \text{ is a suffix of } p_u \}$$

and let $M_{u,w}(X)$ be the set of words of the form $p_{u,i}p_{u,i+1} \ldots p_{u,i+j}$ ($i > 0$) for which

- $p_{u,i-1}$ and $p_{u,i+j}$ are in $L_{u,w}(X)$
- $p_{u,k}$ is not in $L_{u,w}(X)$ for $i \leq k < i + j$.

Our sequence $X$ can be written uniquely as a concatenation of elements in $M_{u,w}(X)$,

$$X = m_{u,1}m_{u,2} \ldots m_{u,i} \in M_{u,w}(X),$$

while $D_u(X)$ admits a unique decomposition as a concatenation

$$D_u(X) = l_{u,1}l_{u,2} \ldots,$$

where each $l_{u,i}$ is a word in the alphabet of $D_u(X)$ which ends in a letter of $\Lambda_{X,u}(L_{u,w}(X))$ and contains no other occurrence of a letter of $\Lambda_{X,u}(L_{u,w}(X))$, and for which there is some letter $a \in \Lambda_{X,u}(L_{u,w}(X))$ such that $al_{u,i}$ occurs in $D_u(X)$. This latter decomposition is completely determined by the sequence $D_u(X)$ and the finite subset $\Lambda_{X,u}(L_{u,w}(X))$ of its alphabet. It is easy to see that

$$D_u(X) = \Lambda_{X,u}(m_{u,1})\Lambda_{X,u}(m_{u,2}) \ldots$$

is one such decomposition, and thus $l_{u,i} = \Lambda_{X,u}(m_{u,i})$ for each $i$. Order $\{ l_{u,i} : i \geq 1 \}$ by the rank of first occurrence in this sequence and let $\Gamma_{X,u,w} : \{ l_{u,i} : i \geq 1 \} \rightarrow \{ 1, 2, \ldots, \text{Card}(M_{u,w}(X)) \}$ be the induced bijection. The above remarks show that as $u$ varies we obtain only finitely many sequences $\Gamma_{X,u,w}(\Lambda_{X,u}(m_{u,1})\Lambda_{X,u}(m_{u,2}) \ldots)$.

Now, the return words of $Y$ to $wu$ are precisely the words $q_{u,i}$ defined by $q_{u,i}w = wnu_i$ and, with this notation,

$$Y = q_{u,1}q_{u,2} \ldots,$$

from which it follows that

$$D_{wu}(Y) = \Gamma_{X,u,w}(\Lambda_{X,u}(m_{u,1}))\Gamma_{X,u,w}(\Lambda_{X,u}(m_{u,2})) \ldots.$$

This shows that $Y$ has only finitely many derived sequences arising from the prefixes $wu$, and of course, only finitely many from shorter prefixes. The lemma is proved.

4. Examples

As an immediate consequence of Theorem 1 one obtains

**Corollary 3.** Let $*$ be a binary operation on $\mathcal{A}$ and let \( \omega = \omega_1\omega_2\omega_3 \ldots \) be a u.p.s. sequence in $\mathcal{A}$. Then the induced sequence of partial products

$$\omega_1, \omega_1 * \omega_2, (\omega_1 * \omega_2) * \omega_3, ((\omega_1 * \omega_2) * \omega_3) * \omega_4, \ldots$$

is u.p.s.
Corollary 3 was used by Ketkar and Zamboni (1999) to show that the set $M(r)$ of real numbers whose base $r$ digit expansion is u.p.s., is closed under multiplication by $Q$.

For another example, consider a primitive substitutive sequence $\omega$ on an alphabet $A$. Fix a positive integer $k$ and define a binary operation $\ast$ on $A \cup \{\triangle_1, \triangle_2, \ldots, \triangle_{k-1}\}$ by setting

$$ a \ast b = \begin{cases} 
\triangle_{i+1} & \text{if } a = \triangle_i \text{ and } i < k-1 \\
\triangle_1 & \text{otherwise.}
\end{cases} $$

Then the sequence of partial products (2) is equal to

$$ \omega_1, \triangle_1, \ldots, \triangle_{k-1}, \omega_{k+1}, \triangle_1, \ldots, \triangle_{k-1}, \omega_{2k+1}, \triangle_1, \ldots, $$

It is straightforward to show that this sequence is recurrent, hence primitive substitutive by Corollary 3. Since the sequence obtained by deleting all occurrences of a given letter from a primitive substitutive sequence is itself primitive substitutive, it follows that the subsequence $(\omega_{nk+1})_{n \geq 0}$ is also primitive substitutive. The same argument applied to the shifted sequences $T^j \omega$ (all of which are primitive substitutive by Lemma 2) yields

**Corollary 4.** Let $j$ and $k$ be positive integers. If $\omega$ is primitive substitutive then so is the subsequence $(\omega_{nk+j})_{n \geq 0}$.

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